JOURNAL OF APPROXIMATION THEORY 22, 46–58 (1978)

Nearly Coconvex Approximation of Order k

DIANE-CLAIRE MYERS[1]

Department of Science and Mathematics, Wesleyan College, Macon, Georgia 32101

Communicated by Carl de Boor

Received May 4, 1976

1. INTRODUCTION

We denote by $C[a, b]$ the class of continuous functions on the interval $[a, b]$, and by $C^1[a, b]$ the class of continuously differentiable functions on that interval. We let $\|f\|$ denote the sup norm of $f \in C[a, b]$ on $[a, b]$, $\omega(f; \cdot)$ denote the modulus of continuity of f on $[a, b]$, and Π_n, $n = 1, 2,\ldots,$ denote the set of algebraic polynomials of degree less than or equal to n.

We define a function $f \in C[a, b]$ to be *convex of order* k, or, more simply, k-*convex* on $[a, b]$ [*concave of order* k, or k-*concave* on $[a, b]$] if its kth order differences are nonnegative [nonpositive] on $[a, b]$; i.e., if

$$\Delta_t^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jt) \geqslant 0 \qquad [\leqslant 0],$$

whenever $x + jt \in [a, b]$, $j = 0, 1, 2,\ldots, k$, and $t > 0$. We note that, according to our definition, a convex function of order 1 is simply a nondecreasing function; a convex function of order 2 is simply an ordinary convex function.

More generally, we define a function $f \in C[a, b]$ to be *piecewise convex of order* k, or, more simply, *piecewise* k-*convex* on $[a, b]$, if we can find a finite sequence of points $\{\beta_i\}$ $(i = 0, 1,\ldots, m)$ with

$$a = \beta_0 < \beta_1 < \cdots < \beta_m = b,$$

such that, within each of the intervals (β_i, β_{i+1}), $i = 0, 1,\ldots, m-1$, $f(x)$ is k-convex or $f(x)$ is k-concave. We call such a set of points $\{\beta_i\}$ $(i = 0, 1,\ldots, m)$ a set of *crucial points* for f.

Note. We note that a set of crucial points need not be unique, for we may arbitrarily assign $f(x)$ to be k-convex or k-concave on any interval $(c, d) \subset [a, b]$ on which $\Delta_t^k f(x) \equiv 0$. In the paper, whenever we take $\{\beta_i\}$ to be a set of crucial points for f, we will assume that a fixed signature has been assigned to $\Delta_t^k f(x)$ in the collection of subintervals $S = \{(\beta_i, \beta_{i+1})\}$ determined by these points.

[1] Current address: Department of Mathematics, Emory University, Atlanta, Georgia 30322.

0021-9045/78/0221-0046$02.00/0

We note also that if $\{\beta_i\}$ is a set of crucial points for f, then $\Delta_t^k f(x)$ need not necessarily alternate in sign on the intervals belonging to the collection S; we observe that, if $k \geqslant 2$, it is possible for a continuous function f to be, say, k-convex on each of two adjacent intervals belonging to S and yet not be k-convex on their union.

Again, in the special cases $k = 1$ and $k = 2$, we refer to "piecewise convex functions of order 1" as being *piecewise monotone* and "piecewise convex functions of order 2" as being *piecewise convex*.

We say that $g(x) \in C[a, b]$ is *co-k-convex* with the piecewise k-convex function f on $[a, b]$ (with respect to the set $\{\beta_i\}$ $(i = 0, 1,..., m)$) if $g(x)$ is k-convex on the intervals (β_i, β_{i+1}) on which $f(x)$ is k-convex and k-concave on the intervals (β_i, β_{i+1}) on which $f(x)$ is k-concave. More generally, if $S = \{I_i\}$ $(i = 1,..., j)$ is a collection of disjoint subintervals of $[a, b]$ on each of which $f(x)$ is k-convex or k-concave, then we say that $g(x)$ is co-k-convex with $f(x)$ on S if $f(x)$ and $g(x)$ share the same k-convexity properties on each interval $I_i \in S$.

In this paper we will be concerned primarily with a type of approximation known as *nearly coconvex approximation of order k*, or, more simply, *nearly co-k-convex approximation*.

DEFINITION. Suppose that $f(x) \in C[a, b]$ is piecewise convex of order k on $[a, b]$ and suppose that $\{\beta_i\}$ $(i = 0, 1,..., m)$ is a set of crucial points for f. A sequence of algebraic polynomials $\{P_n(x)\}$ is said to form a *nearly co-k-convex approximating sequence* to f on $[a, b]$ (with respect to the set $\{\beta_i\}$) if, given any $0 < \epsilon < \delta = \frac{1}{2} \min_i\{\beta_{i+1} - \beta_i\}$, there exists N_ϵ, such that, if $n \geqslant N_\epsilon$, then $P_n(x)$ is co-k-convex with f on $S_\epsilon = \{(\beta_i + \epsilon, \beta_{i+1} - \epsilon)\}$ $(i = 0, 1,..., m - 1)$.

This type of approximation was first considered for the case $k = 1$, and, in fact, in the special cases $k = 1$ and $k = 2$, we refer to such sequences $\{P_n(x)\}$ as being *nearly comonotone* or *nearly coconvex approximating sequences*, respectively.

Note. For the sake of simplicity, throughout the remainder of the paper, whenever we speak of approximating a given piecewise k-convex function in a nearly co-k-convex fashion, we will assume that we have selected, and fixed, a particular set of crucial points for this function and that we are performing our approximation with respect to this particular set of points.

The following result on nearly comonotone approximation is known:

THEOREM A. *Given $f \in C[-1, 1]$, piecewise monotone on $[-1, 1]$, then there exists a nearly comonotone approximating sequence to f, $\{P_n(x)\}$, such that*

$$\|f - P_n\| \leqslant c_1 \omega(f; n^{-1}),$$

where $c_1 > 0$ is an absolute constant.

The notion of nearly comonotone approximation was first studied by Newman *et al.* [8], who showed that Theorem A was valid for a class of functions known as "proper" piecewise monotone functions. The restriction that f be "proper" was later removed by DeVore in [3]. (For additional results on nearly comonotone approximation, see also Roulier [9].)

Theorem A was later extended by Myers and Raymon [6] to the case $k = 2$:

THEOREM B. *Given $f \in C[-1, 1]$, piecewise convex on $[-1, 1]$, there exists a nearly coconvex approximating sequence to f, $\{P_n(x)\}$, such that*

$$\|f - P_n\| \leqslant c_2 \omega(f; n^{-1}),$$

where $c_2 > 0$ is an absolute constant.

In this paper, we will, in our first theorem, extend Theorems A and B to obtain Jackson-order nearly coconvex approximation of order k to piecewise k-convex functions for any positive integer k. In our second theorem we obtain a Jackson-type theorem for nearly coconvex approximation of order k to piecewise k-convex functions belonging to $C^1[-1, 1]$, where k is any positive integer. We now formally state our main results:

Let k be any positive integer.

THEOREM 1. *Given $f \in C[-1, 1]$, piecewise k-convex on $[-1, 1]$, there exists a nearly co-k-convex approximating sequence to f, $\{P_n(x)\}$, such that*

$$\|f - P_n\| \leqslant c_k \omega(f; n^{-1}),$$

where $c_k > 0$ is a constant depending only on k.

THEOREM 2. *Given $f \in C^1[-1, 1]$, piecewise k-convex on $[-1, 1]$, there exists a nearly co-k-convex approximating sequence to f, $\{P_n(x)\}$, such that*

$$\|f - P_n\| \leqslant c_k{}^* n^{-1} \omega(f; n^{-1}),$$

where $c_k{}^* > 0$ is a constant depending only on k.

2. PRELIMINARY LEMMAS

In proving our theorems we will use polynomials formed by convolution with an algebraic kernel. In particular, we will use the nth DeVore-j or D-j kernels, defined to be $V_{n,j}(t)$, where

$$V_{n,j}(t) = v_{n,j} \left[\frac{P_{2n}(t)}{(t^2 - \alpha_{1,n}^2) \cdots (t^2 - \alpha_{j,n}^2)}\right]^2, \tag{1}$$

where $P_{2n}(t)$ is the Legendre polynomial of degree $2n$, $\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{j,n}$ are the j smallest positive zeros of $P_{2n}(t)$, respectively, and $v_{n,j}$ is a normalizing constant chosen so that

$$\int_{-1}^{1} V_{n,j}(t)\, dt = 1.$$

Defining the nth D-j polynomial of $f \in C[-\frac{1}{2}, \frac{1}{2}]$ to be

$$Q_{n,j}(f; x) = \int_{-1/2}^{1/2} V_{n,j}(t-x) f(t)\, dt,$$

then $Q_{n,j}(f; x) \in \Pi_{4(n-j)}$ for each $n \geqslant j$.

It is shown in [2, Chap. 6] that if $0 < \delta < \frac{1}{2}$, then

$$\|f - Q_{n,j}(f)\|_{[-\delta,\delta]} \leqslant c_\delta \omega(f; n^{-1}), \tag{2}$$

where c_δ is a constant depending only on j and δ. If, in addition, $f \in C^1[-\frac{1}{2}, \frac{1}{2}]$, then it is also shown in [2] that

$$\|f - Q_{n,j}(f)\|_{[-\delta,\delta]} \leqslant c_\delta' n^{-1} \omega(f'; n^{-1}), \tag{3}$$

where c_δ' is a constant depending only on j and δ.

In [6], we obtained certain estimates on the D-1 kernel. These proofs may easily be modified to yield the following important estimates for the general D-j kernels $V_{n,j}(x)$:

LEMMA 1. *Let $\epsilon > 0$ be given and let $1 > \nu > 0$ be specified. Then there exists a positive constant $d(j)$, depending only on ν and j, such that, for all $n \geqslant 2(j+1)/\epsilon$, if $\nu \geqslant |x| \geqslant \epsilon$, then*

$$0 \leqslant V_{n,j}(x) \leqslant d(j)\, n^{-4j+1} \epsilon^{-4j}.$$

The following five lemmas are all well-known results:

LEMMA 2 (Second Mean Value Theorem [11, p. 381]). *If $f(x)$ is a bounded, monotonic function on $[a, b]$ and if $\phi(x)$ is integrable over $[a, b]$, then there is a number $\xi \in [a, b]$ such that*

$$\int_a^b \phi(x) f(x)\, dx = f(a+0) \int_a^\xi \phi(x)\, dx + f(b-0) \int_\xi^b \phi(x)\, dx.$$

(Here, $f(a+0)$ and $f(b-0)$ denote the right-hand limit of f at a and the left-hand limit of f at b, respectively.)

LEMMA 3 [10, pp. 108–109]. *If $f(x)$ is convex on (a, b), then f is absolutely continuous on each closed subinterval of (a, b). The right- and left-hand derivatives of f exist at each point of (a, b) and are equal to each other except on a countable set. The left- and right-hand derivatives are monotone increasing functions and at each point the left-hand derivative is less than or equal to the right-hand derivative.*

LEMMA 4 [1]. *If $f(x)$ is k-convex on the interval $[a, b]$, for some $k \geqslant 2$, then $f, f', f'', \ldots, f^{(k-2)}$ exist and are continuous in the open interval (a, b).*

LEMMA 5 (Markov's Inequality [7, p. 141]). *If a polynomial $p(x) \in \Pi_n$ satisfies, for each $a \leqslant x \leqslant b$,*

$$|p(x)| \leqslant M,$$

then $p'(x)$ satisfies, for each $a \leqslant x \leqslant b$,

$$|p'(x)| \leqslant 2Mn^2/(b - a).$$

We prove two final lemmas:

LEMMA 6. *Suppose that, for $k \geqslant 2$, $f(x)$ is k-convex on the interval $[a - \epsilon, b + \epsilon]$ for some $\epsilon > 0$. Define, for each n,*

$$w_n(f; x) = w_n(x) = \int_a^b V_n(x - t) f(t)\, dt,$$

where $V_{n,k}(t) = V_n(t)$ is the D-k kernel defined by (1). Let $x \in [a, b]$. Then there exists $\xi_x \in [a, b]$ such that

$$w_n^{(k)}(x) = \sum_{i=1}^{k} [-f^{(i-1)}(b - 0)\, V_n^{(k-i)}(x - b) + f^{(i-1)}(a + 0)\, V_n^{(k-i)}(x - a)] + V_n(x - \xi_x)[f^{(k-1)}(b - 0) - f^{(k-1)}(a + 0)]. \tag{4}$$

Proof. We note that, by Lemma 4, $f \in C^{k-2}[a, b]$, and, by hypothesis, that $f^{(k-2)}(x)$ is a convex function on $(a - \epsilon, b + \epsilon)$ and hence has the properties guaranteed by Lemma 3. Integrating $w_n(x)$ by parts, then differentiating with respect to x, we get that

$$w_n'(x) = -f(b)\, V_n(x - b) + f(a)\, V_n(x - a) + \int_a^b V_n(x - t) f'(t)\, dt.$$

Repeating this procedure $(k - 2)$ more times gives us that

$$w_n^{(k-1)}(x) = \sum_{i=2}^{k} [-f^{(i-2)}(b)\, V_n^{(k-i)}(x - b) + f^{(i-2)}(a)\, V_n^{(k-i)}(x - a)] + I(x), \tag{5}$$

where $I(x) = \int_a^b V_n(x - t) f^{(k-1)}(t)\, dt$.

Since $V_n(x - t)$ is an algebraic polynomial in $(x - t)$, we may, with respect to x, differentiate $I(x)$ inside the integral sign. Differentiating, then, each of the other terms in (5) with respect to x, applying Lemma 2 to $I'(x)$, and then using the Fundamental Theorem of Calculus to evaluate the expressions

$$\int_a^{\xi_x} V_n'(x - t)\, dt, \qquad \int_{\xi_x}^b V_n'(x - t)\, dt,$$

we get, for the derivative of (5), expression (4).

LEMMA 7. *Let $f(x)$ be k-convex on the interval $[a, b]$ for some $k \geqslant 2$. Let $0 < \epsilon < (b - a)/2k$ be given, and assume that ϵ has been chosen so that $f(x)$ has no changes in j-convexity, $j = 1, 2,..., k - 1$, within the intervals*

$$I_\epsilon = [a + \epsilon, a + (2k - 1)\epsilon] \quad \textit{and} \quad J_\epsilon = [b - (2k - 1)\epsilon, b - \epsilon].$$

We know, by Lemma 4, that $f', f'',..., f^{(k-2)}$ exist and are absolutely continuous on I_ϵ and J_ϵ. We claim that

$$|f^{(j)}(a + k\epsilon + 0)| \leqslant \epsilon^{-j}\omega(f; \epsilon)$$

and

$$|f^{(j)}(b - k\epsilon - 0)| \leqslant \epsilon^{-j}\omega(f; \epsilon), \qquad j = 1, 2,..., k - 1.$$

Proof. It suffices to show the lemma for the point $\alpha = a + k\epsilon$, and under the assumption that $f^{(j)}(x)$ is nonnegative (and nondecreasing) on I_ϵ for each $j = 1, 2,..., k - 1$. The proof will be carried out by induction on j.

Consider first the case $j = 1$. Since $f(x)$ is absolutely continuous on I_ϵ, we have

$$\begin{aligned}\omega(f; \epsilon) &\geqslant f(\alpha + (k - 1)\epsilon) - f(\alpha + (k - 2)\epsilon)\\ &= \int_{\alpha+(k-2)\epsilon}^{\alpha+(k-1)\epsilon} f'(t)\, dt\\ &\geqslant \epsilon \cdot f'(\alpha + (k - 2)\epsilon)\\ &\geqslant \epsilon \cdot f'(\alpha).\end{aligned}$$

Suppose now that the result holds for $j = 1, 2,..., i$. In particular, suppose that we have shown, for $j = 1, 2,..., i$, that

$$\epsilon^{-j}\omega(f; \epsilon) \geqslant f^{(j)}(\alpha + (k - j - 1)\epsilon).$$

Then, using our assumptions on the $f^{(j)}(x)$, we have

$$\begin{aligned}\epsilon^{-i}\omega(f; \epsilon) &\geqslant f^{(i)}(\alpha + (k - i - 1)\epsilon)\\ &\geqslant f^{(i)}(\alpha + (k - i - 1)\epsilon) - f^{(i)}(\alpha + (k - i - 2)\epsilon)\\ &= \int_{\alpha+(k-i-2)\epsilon}^{\alpha+(k-i-1)\epsilon} f^{(i+1)}(t)\, dt\\ &\geqslant \epsilon \cdot f^{(i+1)}(\alpha + (k - i - 2)\epsilon)\\ &\geqslant \epsilon \cdot f^{(i+1)}(\alpha).\end{aligned}$$

Now, the lemma has been shown under certain very specific assumptions on the $f^{(j)}(x)$; in the event that the signs of the $f^{(j)}$ on I_ϵ are mixed, then, clearly, by considering, for the appropriate j, intervals of the form $[\alpha - (k - j)\epsilon, \alpha - (k - j - 1)\epsilon] \subset [a + \epsilon, \alpha]$, the proof of Lemma 7 can easily be modified to yield the general result.

3. Proofs of the Theorems

Proof of Theorem 1. Without loss of generality we prove the result for f piecewise k-convex on the interval $[0, \frac{1}{8}]$ and for $k \geqslant 2$. It suffices to show the existence of a sequence $\{P_n(x)\}$ $(n \geqslant N)$ satisfying the conclusions of Theorem 1, with $P_n(x) \in \Pi_{ln}$ for each such n, l a fixed positive integer. We let the crucial points of f occur at the points

$$0 = \beta_0 < \beta_1 < \cdots < \beta_m = \tfrac{1}{8};$$

we let $0 < \delta < \frac{1}{2} \min_i\{\beta_{i+1} - \beta_i\}$ be chosen so that $f(x)$ has no changes in j-convexity, $j = 1, 2,..., k$, within the intervals $(\beta_i - \delta, \beta_i)$, $i = 1, 2,..., m$, and $(\beta_i, \beta_i + \delta)$, $i = 0, 1,..., m - 1$. Without loss of generality, we assume that $f(0) = 0$. Throughout the remainder of the paper we will take $c_1(k)$, $c_2(k)$,..., $d_1(k)$, $d_2(k)$,..., to be positive constants depending only on k. All norms, unless otherwise stated, will be taken on the interval $[0, \frac{1}{8}]$.

The proof will be carried out in two steps. In the first step we show that, given $0 < \epsilon < \frac{1}{2}\delta$, there exists a sequence of polynomials $\{P_{n,\epsilon}(x)\}$, corresponding to ϵ, and satisfying:

(i) $P_{n,\epsilon}(x) \in \Pi_{4(n-k)}$ for $n \geqslant m + k - 1$; (6)

(ii) For n sufficiently large, $P_{n,\epsilon}(x)$ is a "good" approximation to f on $[0, \frac{1}{8}]$; (7)

(iii) $P_{n,\epsilon}(x)$ is co-k-convex with $f(x)$ on the intervals belonging to $S_\epsilon = \{(\beta_i + \epsilon, \beta_{i+1} - \epsilon)\}$ $(i = 0, 1,..., m - 1)$ for n sufficiently large. (8)

In the second step, by choosing an appropriate sequence $\{\epsilon_n\} \to 0$, we will select from the collection of sequences $\{P_{n,\epsilon_n}(x)\}$ a single nearly co-k-convex approximating sequence.

Step 1. We extend $f(x)$ to $[-\frac{1}{2}, \frac{1}{2}]$ (and we call the extension $f(x)$ also), by defining

$$\begin{aligned} f(x) &= f(0), && -\tfrac{1}{2} \leqslant x < 0, \\ &= f(x), && 0 \leqslant x \leqslant \tfrac{1}{8}, \\ &= f(\tfrac{1}{8}), && \tfrac{1}{8} < x \leqslant \tfrac{1}{2}. \end{aligned}$$

For each $n \geqslant k$, we let $Q_n(f; x) = Q_n(x)$ be the nth D-k polynomial of the (extended) function $f(x)$. Then, by (2), we have, (on $[0, \frac{1}{8}]$),

$$\| f - Q_n \| \leqslant c_1(k)\, \omega(f; n^{-1}). \tag{9}$$

Let $0 < \epsilon < \frac{1}{2}\delta$ be given. Define, for each $i = 0, 1,..., m - 1$, $J_i = (\beta_i + \epsilon, \beta_{i+1} - \epsilon)$. We will show, by estimating $Q_n^{(k)}(x)$ in each of the intervals J_i, that $Q_n(x)$ is not "far" from being co-k-convex with f in these intervals. We will then construct our $P_{n,\epsilon}(x)$ by adding to each $Q_n(x)$ a suitable "correcting" polynomial.

Let $x \in J_i$ for some $i = 0, 1,..., m - 1$, where we assume that $f(x)$ is k-convex on J_i. Let $a_i = \beta_i + \frac{1}{2}\epsilon$; $b_i = \beta_{i+1} - \frac{1}{2}\epsilon$. We may write

$$\begin{aligned} Q_n(x) &= \int_{-1/2}^{1/2} V_n(x - t) f(t)\, dt \\ &= I_1(x) + I_2(x) + I_3(x), \end{aligned}$$

where

$$I_{1,n}(x) = I_1(x) = \int_{-1/2}^{a_i} V_n(x - t) f(t)\, dt,$$

$$I_{2,n}(x) = I_2(x) = \int_{a_i}^{b_i} V_n(x - t) f(t)\, dt,$$

and

$$I_{3,n}(x) = I_3(x) = \int_{b}^{1/2} V_n(x - t) f(t)\, dt.$$

Then

$$Q_n^{(k)}(x) = I_1^{(k)}(x) + I_2^{(k)}(x) + I_3^{(k)}(x). \tag{10}$$

Since $|x - t| \geqslant \frac{1}{2}\epsilon$ whenever $x \in J_i$ and $t \notin (a_i, b_i)$, it follows from Lemma 1 that for all such t and for all $n \geqslant 4(k+1)/\epsilon = N_k$, we have

$$\max_{x \in J_i} |V_n(x-t)| \leqslant d_1(k)\, n^{-4k+1}(\tfrac{1}{2}\epsilon)^{-4k}. \tag{11}$$

Since, by assumption, $f(0) = 0$, it follows from the sublinearity property of $\omega(f; \cdot)$ that

$$\|f\|_{[-\frac{1}{2},\frac{1}{2}]} \leqslant \omega(f; 1) \leqslant n\omega(f; n^{-1}).$$

Thus, for all $n \geqslant N_k$ and for any $x \in J_i$, we have that

$$\begin{aligned} |I_1(x) + I_3(x)| &\leqslant d_1(k)\, \|f\|\, n^{-4k+1}(\tfrac{1}{2}\epsilon)^{-4k} \\ &\leqslant d_1(k)\, n^{-4k+2}\omega(f; n^{-1})(\tfrac{1}{2}\epsilon)^{-4k}. \end{aligned} \tag{12}$$

Applying Lemma 5 to the polynomial $I_1(x) + I_3(x)$ on J_i, and using (12), we get that, for all $n \geqslant N_k$ and for any $x \in J_i$,

$$|I_1^{(k)}(x) + I_3^{(k)}(x)| \leqslant d_2(k)\, n^{-2k+2}\omega(f; n^{-1})(\tfrac{1}{2}\epsilon)^{-4k}. \tag{13}$$

Also, applying Lemma 5 to the polynomials $V_n(x - a_i)$ and $V_n(x - b_i)$, respectively on J_i, and using (11), we have that for $n \geqslant N_k$ and for any $x \in J_i$,

$$|V_n^{(j)}(x - a_i)| \leqslant d_2(k)\, n^{-4k+2j+1}(\tfrac{1}{2}\epsilon)^{-4k}, \qquad j = 0, 1,..., k-1, \tag{14}$$

and, similarly,

$$|V_n^{(j)}(x - b_i)| \leqslant d_2(k)\, n^{-4k+2j+1}(\tfrac{1}{2}\epsilon)^{-4k}, \qquad j = 0, 1,..., k-1. \tag{15}$$

By Lemma 7 (with $\epsilon/2k$ playing the role of ϵ), we have that

$$|f^{(j)}(a_i + 0)| \leqslant (k)^j(\tfrac{1}{2}\epsilon)^{-j}\, \omega(f; \epsilon), \qquad j = 1, 2,..., k-1, \tag{16}$$

and

$$|f^{(j)}(b_i - 0)| \leqslant (k)^j(\tfrac{1}{2}\epsilon)^{-j}\, \omega(f; \epsilon), \qquad j = 1, 2,..., k-1. \tag{17}$$

Hence, using (14)–(17), together with the bound we have on $\|f\|$, plus the fact that

$$\omega(f; \epsilon) \leqslant n\omega(f; n^{-1}),$$

we have that for all $n \geqslant N_k$,

$$|f^{(j-1)}(a_i + 0)\, V_n^{(k-j)}(x - a_i)| \leqslant d_3(k)\, n^{-2k}\omega(f; n^{-1})(\tfrac{1}{2}\epsilon)^{-5k}, \qquad j = 1, 2,..., k. \tag{18}$$

Similarly, we have for all $n \geqslant N_k$,

$$|-f^{(j-1)}(b_i - 0)\, V_n^{(k-j)}(x - b_i)| \leqslant d_3(k)\, n^{-2k}\omega(f; n^{-1})(\tfrac{1}{2}\epsilon)^{-5k}, \quad j = 1, 2,..., k. \tag{19}$$

Using, now, Lemma 6, with a_i and b_i playing the roles of a and b, respectively, and summing over j, $j = 1, 2,..., k$, we can find $\xi_x \in [a_i, b_i]$ such that $I_2^{(k)}(x)$ is given by (4). Using the fact that $f^{(k-1)}$ is nondecreasing on $[a_i, b_i]$, together with the nonnegativity of $V_n(x)$, we have that

$$V_n(x - \xi_x)[f^{(k-1)}(b_i - 0) - f^{(k-1)}(a_i + 0)] \geqslant 0. \tag{20}$$

Thus, using (18)–(20) in (4), we have that, for all $n \geqslant N_k$,

$$I_2^{(k)}(x) \geqslant -2kd_3(k)\, n^{-2k}\omega(f; n^{-1})(\tfrac{1}{2}\epsilon)^{-5k}. \tag{21}$$

Finally, using estimates (13) and (21) in (10), we may assert that for $x \in J_i$, where $f(x)$ is k-convex on J_i, and for $n \geqslant N_k$,

$$Q_n^{(k)}(x) \geqslant -d_4(k)\, n^{-2k+2}\omega(f; n^{-1})(\tfrac{1}{2}\epsilon)^{-5k}. \tag{22}$$

Similarly, for $x \in J_i$, where $f(x)$ is k-concave on J_i, and for $n \geqslant N_k$, we have

$$Q_n^{(k)}(x) \leqslant d_4(k)\, n^{-2k+2}\omega(f; n^{-1})(\tfrac{1}{2}\epsilon)^{-5k}. \tag{23}$$

Let p be the number of points β_i for which one of the following occurs:

(i) J_{i-1} is a k-concave interval and J_i is a k-convex interval,

or

(ii) J_{i-1} is a k-convex interval and J_i is a k-concave interval.

Then $0 \leqslant p \leqslant m - 1$. Let $z_1 < z_2 < \cdots < z_p$ be those points, if any, belonging to $\{\beta_i\}$ $(i = 0, 1,..., m)$ which satisfy either (i) or (ii). Define, for each n,

$$\gamma_n = d_4(k)\, n^{-2k+2}\omega(f; n^{-1})(\tfrac{1}{2}\epsilon)^{-5k-p}. \tag{24}$$

If $p = 0$, we define

$$P_{n,\epsilon}(x) = Q_n(x) + \sigma\gamma_n x^k,$$

where

$$\begin{aligned} \sigma &= 1, && \text{if each } J_i \text{ is } k\text{-convex, } i = 0, 1,..., m-1,\\ &= -1, && \text{if each } J_i \text{ is } k\text{-concave, } i = 0, 1,..., m-1. \end{aligned}$$

Suppose that $p \geqslant 1$. We assume, without loss of generality, that p is odd and that z_1 satisfies (i). We define

$$P_{n,\epsilon}(x) = Q_n(x) + \gamma_n \int_0^x \int_0^{t_{k-1}} \cdots \int_0^{t_1} P(t)\, dt\, dt_1 \cdots dt_{k-1},$$

where $P(t) = \prod_{i=1}^{p} (t - z_i)$.

We see immediately that the polynomials $\{P_{n,\epsilon}(x)\}$ satisfy (6). It follows from (22)–(23) that they satisfy (8). Since

$$\| f - P_{n,\epsilon} \| \leqslant \| f - Q_n \| + \gamma_n, \tag{25}$$

we see from (9) and from (24) that (7) is satisfied.

Step 2. We now select the desired sequence $\{P_n(x)\}$: We define $t = (5k + p)^{-1}$. For each n satisfying

$$2n^{-t} < \tfrac{1}{2}\delta, \tag{26}$$

we define

$$\epsilon_n = 2n^{-t}, \tag{27}$$

and we take

$$P_n(x) \equiv P_{n,\epsilon_n}(x).$$

Then $\{P_n(x)\}$ is a nearly co-k-convex approximating sequence to f on $[0, \frac{1}{8}]$, for, choosing $\epsilon < \frac{1}{2}\delta$, then, by Step 1, $P_n(x)$ is co-k-convex with $f(x)$ on the intervals $(\beta_i + \epsilon_n, \beta_{i+1} - \epsilon_n)$, $i = 0, 1, \ldots, m - 1$, hence on the intervals $(\beta_i + \epsilon, \beta_{i+1} - \epsilon)$, $i = 0, 1, \ldots, m - 1$, for all n satisfying (26) and also

$$n \geqslant \max\{2(k + 1)\, n^t, (\tfrac{1}{2}\epsilon)^{-1/t}\}.$$

Finally, using (9), (24), and (27) in (25), we have that

$$\begin{aligned} \| f - P_n \| &= \| f - P_{n,\epsilon_n} \| \\ &\leqslant [c_1(k) + d_4(k)\, n^{-2k+3}]\, \omega(f; n^{-1}). \qquad \text{Q.E.D.} \end{aligned}$$

Proof of Theorem 2. Again, without loss of generality, we work on the interval $[0, \frac{1}{8}]$. Using (3), the proof of Theorem 2 is almost identical to that of Theorem 1. We extend $f(x) \in C^1[0, \frac{1}{8}]$ to $[-\frac{1}{2}, \frac{1}{2}]$ in a continuously differentiable fashion by defining the (extended) function $f(x)$ on the intervals $[-\frac{1}{2}, 0]$ and $[\frac{1}{8}, \frac{1}{2}]$, respectively, to be those line segments which interpolate $f(x)$ at the points $x = 0$ and $x = \frac{1}{8}$ with respective slopes $f'(0)$ and $f'(\frac{1}{8})$.

Without loss of generality we may assume, for all k, that $f(0) = 0$, and that there exists $\mu \in [-\frac{1}{2}, \frac{1}{2}]$ such that $f'(\mu) = 0$. (For if f' were, say, strictly positive on $[-\frac{1}{2}, \frac{1}{2}]$, then, letting $m = \min f'(x)$, $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$, we need only consider the function $g(x) = f(x) - mx$ on the interval $[-\frac{1}{2}, \frac{1}{2}]$.) Under these assumptions we get, on the interval $[-\frac{1}{2}, \frac{1}{2}]$,

$$\|f\| \leqslant \|f'\| \leqslant \omega(f'; 1) \leqslant n\omega(f'; n^{-1}). \tag{28}$$

If $k \geqslant 2$, we may show, under the hypotheses of Lemma 7, using the same notation, and employing a proof which is almost identical to that of Lemma 7, that

$$|f^{(j)}(a + k\epsilon + 0)| \leqslant (\epsilon)^{-j+1}\,\omega(f'; \epsilon)$$

and

$$|f^{(j)}(b - k\epsilon - 0)| \leqslant (\epsilon)^{-j+1}\,\omega(f'; \epsilon), \qquad j = 2, 3,..., k - 1. \tag{29}$$

Now, using the estimates (3), (28), and (29), and working, as before, with the D-k polynomials of the (extended) function f in the case $k \geqslant 2$ (and with the D-2 polynomials of f in the case $k = 1$), our result follows almost exactly as before. Q.E.D.

4. Concluding Remarks

We note that Theorems A and B quoted in the Introduction are actually true in a stronger form. Namely, given a piecewise monotone or a piecewise convex function $f \in C[a, b]$, it is possible to show the existence of respective "strong" nearly comonotone or "strong" nearly coconvex sequences $\{P_n(x)\}$ which approximate f to within the Jackson-order estimates, where we define the term "strong nearly co-k-convex approximating sequence" in the following way:

Definition. A sequence of algebraic polynomials $\{P_n(x)\}$ is said to form a *strong nearly co-k-convex approximating sequence* to a piecewise k-convex function $f \in C[a, b]$ (with respect to the crucial points $a = \beta_0 < \beta_1 < \cdots < \beta_m = b$), if, given any $0 < \epsilon < \delta$ ($\delta = \frac{1}{2}\min_i\{\beta_{i+1} - \beta_i\}$), there exists N_ϵ such that if $n \geqslant N_\epsilon$, the polynomials $P_n(x)$ are co-k-convex with f in the intervals $\{(a, \beta_1 - \epsilon), (\beta_i + \epsilon, \beta_{i+1} - \epsilon), (\beta_m + \epsilon, b)\}$ ($i = 1, 2,..., m - 2$).

These stronger theorems on nearly comonotone and nearly coconvex approximation yield, in particular, as corollaries, the Lorentz–Zeller results on monotone approximation to monotone functions [4, 5], as well as the Lorentz–Zeller analogs for convex approximation to convex functions [6]. We remark here that the "strong" versions of Theorems 1 and 2 can be shown as well for the case $k = 3$. The proofs of these stronger theorems, along with several related results, will appear in a subsequent paper.

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